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CHAPTER

CALCULUS - II

LEARNING OBJECTIVES

- Functions of more than one variable, limit, continuity partial derivatives, variable to be treated as constant. Total derivative, composite function.
- Differential equations, Exact Differential Equation. Inexact differential equation and solution of Exact differential equation.
- Integrating Factor, Different Methods to obtain Integrating Factor with Examples
- Lagrange's Method of multiplier using constraints and its applications.

3.1 FUNCTIONS OF MORE THAN ONE VARIABLE

We are quite familiar with quantities which depend on two or more variables. As for example the area of a triangle depends on its base and height, area of a rectangle depends on its length and breadth. Similarly area of ellipse depends on semimajor axis a and semiminor axis b . Also extending the idea to three dimensional cases, such as the volume of a parallelepiped depends on three variables length, breadth and height.

Hence we can say mathematically that z is called a function of two variables x and y if z has a unique value (definite value) for every pair of value of x and y . It is denoted by $z = f(x, y)$.

Here x and y are called Independent variables while z is called the dependent variable.

The set of points (x, y) in the xy plane for which $f(x, y)$ is defined is called the domain of the function. The set of corresponding values of z is called the range of the function.

Limits : If $z = f(x, y)$ be a function of two variables then it is said to have a limit l as $x \rightarrow a$, $y \rightarrow b$ if and only if limit l is independent of the path when $x \rightarrow a$, $y \rightarrow b$. It is denoted by

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = l$$

In terms of δ neighbourhood of (a, b) there exists a given real number $\epsilon > 0$ however small, we have a real number $\delta > 0$ such that

$$|f(x, y) - l| < \epsilon \text{ for } |x - a| < \delta \text{ and } |y - b| < \delta$$

then the real, finite number l is called the limit of the function $f(x, y)$ as $(x, y) \rightarrow (a, b)$.
 the limit of $z = f(x, y)$ exists only when the same value is obtained for the limit along any path
 in xy plane as $x \rightarrow a, y \rightarrow b$ or along $y \rightarrow b, x \rightarrow a$.

Results : If $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = l$ $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} g(x, y) = m$

Then 1. $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} (f \pm g) = l \pm m$ 2. $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} (f \cdot g) = l \cdot m$ 3. $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} (f/g) = l/m$ if $m \neq 0$

Continuity : The function $f(x, y)$ is said to be continuous at a point (a, b) if

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = f(a, b).$$

A function is said to be continuous in a domain if it is continuous at every point of domain
 But if the function is not continuous at a point then it is said to be discontinuous at that point.

Example. 1 : Find the value of $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{y^2 - x^2}$

Ans. $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{xy}{y^2 - x^2} = \lim_{y \rightarrow 0} 0 = 0$

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{xy}{y^2 - x^2} = \lim_{x \rightarrow 0} 0 = 0$$

Along the line $y = mx$, we get

$$\lim_{x \rightarrow 0} \frac{(x)(mx)}{(mx)^2 - x^2} = \lim_{x \rightarrow 0} \frac{mx^2}{x^2(m^2 - 1)} = \frac{m}{m^2 - 1}$$

Which is different for different value of m . So the limit does not exist.

Example. 2 : Find the value of $\lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} \frac{2x^2y}{x^2 + y^2 + 1}$

Ans. $\lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} \frac{2x^2y}{x^2 + y^2 + 1} = \lim_{x \rightarrow 1} \frac{2x^2 \cdot 2}{x^2 + 4 + 1} = \lim_{x \rightarrow 1} \frac{4x^2}{x^2 + 5} = \frac{4}{6} = \frac{2}{3}$

$$\lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} \frac{2x^2y}{x^2 + y^2 + 1} = \lim_{y \rightarrow 2} \frac{2y}{y^2 + 2} = \frac{4}{6} = \frac{2}{3}$$

So the value of limit is $2/3$.

Example. 3 : Test the continuity at the function at origin given by

$$f(x, y) = \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} \quad \text{for } x \neq 0, y \neq 0$$

$$= 0 \quad \text{for } x = 0, y = 0$$

$$\text{Ans. } \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y) = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \lim_{y \rightarrow 0} \frac{-y^2}{y} = 0$$

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y) = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \lim_{x \rightarrow 0} \frac{x^2}{x} = 0$$

Along the line $y = mx$, we have

$$\lim_{\substack{y=mx \\ x \rightarrow 0}} f(x, y) = \lim_{\substack{y=mx \\ x \rightarrow 0}} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \lim_{x \rightarrow 0} \frac{x^2 - m^2 x^2}{\sqrt{x^2 + m^2 x^2}} = \lim_{x \rightarrow 0} \frac{x^2(1 - m^2)}{x\sqrt{1 + m^2}} = 0$$

Hence the limit along any path is same and exists having a common value.

i.e., $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{\sqrt{x^2 - y^2}} = 0$ which is also equal to $f(0, 0)$ as $f(0, 0) = 0$. Hence the given

function is continuous at the origin.

Example.4 : Show that the function $f(x, y) = x^2 + 2y$ ($x, y) \neq (1, 2)$
 $= 0$ ($x, y) = (1, 2)$

is discontinuous at $(1, 2)$

$$\text{Ans. } \lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} f(x, y) = \lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} (x^2 + 2y) = \lim_{y \rightarrow 2} (1 + 2y) = 5$$

Also $\lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} (x^2 + 2y) = \lim_{x \rightarrow 1} (x^2 + 4) = 5$. But $f(x, y) = 0$ at $x = 1, y = 2$

Hence $\lim_{(x,y) \rightarrow (1,2)} f(x, y) \neq f(a, b)$. Hence the function is discontinuous at $(1, 2)$

PARTIAL DERIVATIVES

A partial derivative of function of several variable is obtained by taking the ordinary derivative with respect to one of the variables keeping the remaining variables constant.

If $z = f(x, y)$ be a function of two variables x and y , then if we treat y as constant and vary x only then derivative of z with respect to x keeping y as constant is called partial derivative of z with

respect to x . It is denoted by any one of the symbols such as $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $f'_x(x, y)$ or $D_x f$

$$\text{Hence } \frac{\partial z}{\partial x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Also derivative of z with respect to y , keeping x constant is called partial derivative of z with respect to y and is represented by

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y} \text{ or } f'_y(x, y) \text{ or } D_y f$$

$$\text{Such that } \frac{\partial z}{\partial y} = \lim_{\delta y \rightarrow 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

If $z = f(x, y)$ then $\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2}$ which is also represented $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}

Similarly $\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial y \partial x}$ which is represented as $\frac{\partial^2 f}{\partial y \partial x}$ or f_{yx} .

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{xy}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}$$

These are second order derivatives of partial Differentiation. It is noted that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

Example - 1 :

Find the First order partial derivatives of $u = e^x \sin y$

Ans. $u = e^x \sin y \Rightarrow u_x = \frac{\partial u}{\partial x} = e^x \sin y, u_y = \frac{\partial u}{\partial y} = e^x \cos y$

Example - 2 :

If $z = \ln(x + \sqrt{x^2 - y^2})$ Find first order partial derivatives.

Ans. $z = \ln(x + \sqrt{x^2 - y^2}) = \ln \left\{ x + (x^2 - y^2)^{\frac{1}{2}} \right\}$

$$\therefore \frac{\partial z}{\partial x} = \frac{1}{x + \sqrt{x^2 - y^2}} \left[1 + \frac{1}{2}(x^2 - y^2)^{\frac{1}{2}-1} (2x) \right]$$

$$\text{or } \frac{\partial z}{\partial x} = \frac{1}{x + \sqrt{x^2 - y^2}} \left[1 + x(x^2 - y^2)^{-\frac{1}{2}} \right]$$

$$\text{or } \frac{\partial z}{\partial x} = \frac{1 + x(x^2 - y^2)^{-\frac{1}{2}}}{x + \sqrt{x^2 - y^2}}$$

$$\text{Also } \frac{\partial z}{\partial y} = \frac{1}{x + \sqrt{x^2 - y^2}} \left[\frac{1}{2}(x^2 - y^2)^{\frac{1}{2}-1} (-2y) \right] = \frac{-y(x^2 - y^2)^{-\frac{1}{2}}}{x + \sqrt{x^2 - y^2}}$$

Example - 3 :

If $u = \log(x^2 + y^2) + \tan^{-1}(y/x)$ Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Ans. $u = \log(x^2 + y^2) + \tan^{-1}(y/x)$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ The given eqⁿ is exact equation.

The solution is given by $\int_{y \text{ const}} M dx + \int (\text{terms of } N \text{ not containing } x) dy = C$

$$\text{i.e., } \int_{y \text{ const}} (e^y + 1) \cos x dx + \int 0 dy = C \text{ i.e., } e^y \sin x + \sin x = C$$

$$\text{i.e., } (e^y + 1) \sin x = C.$$

3.3 INTEGRATING FACTOR

When the differential equation $M'(x, y) dx + N'(x, y) dy = 0$ is not exact then it can be made exact by multiplying it by suitable factor called Integrating Factor. (I.F)

The various Rules for finding integrating Factors are as follows :

1. Inspection Method

I.F can be obtained sometimes by inspection based on experience and regrouping the terms of equation. For this purpose the student should be familiar with differentials listed below.

$$1. x dy + y dx = d(xy)$$

$$6. \frac{y dx - x dy}{x^2 y^2} = d\left(\frac{1}{xy}\right)$$

$$2. \frac{y dx - x dy}{y^2} = d\left(\frac{x}{y}\right)$$

$$7. \frac{ye^x dx - e^x dy}{y^2} = d\left(\frac{e^x}{y}\right)$$

$$3. \frac{x dy - y dx}{x^2} = d\left(\frac{y}{x}\right)$$

$$8. \frac{2x dx + 2y dy}{x^2 + y^2} = d[\log(x^2 + y^2)]$$

$$4. \frac{y dx - x dy}{x^2 + y^2} = d\left(\tan^{-1} \frac{x}{y}\right)$$

$$9. \frac{2xy dx - x^2 dy}{y^2} = d\left(\frac{x^2}{y}\right)$$

$$5. \frac{y dx - x dy}{xy} = d\left(\log \frac{x}{y}\right)$$

$$10. \frac{2xy^2 dx - 2yx^2 dy}{y^4} = d\left(\frac{x^2}{y^2}\right)$$

Example - 1 : Solve $y dx - x dy = xy^3 dy$

$$\text{Ans. } y dx - x dy = xy^3 dy \text{ i.e. } \frac{y dx - x dy}{xy} = y^2 dy$$

$$\text{i.e., } d\left(\ln \frac{x}{y}\right) = y^2 dy \text{ then Integrating we get } \ln \frac{x}{y} = \frac{y^3}{3} + C$$

Example - 2 : Solve $(1 + xy)y dx + (1 - xy)x dy = 0$

Ans. The given eqⁿ is written as

$$y dy + xy^2 dx + x dy - x^2 y dy = 0$$

$$\text{or } y dx + x dy + xy^2 dx - x^2 y dy = 0$$

$$\text{or } d(xy) + xy^2 dx - x^2 y dy = 0$$

$$\text{i.e., } \frac{d(xy)}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

So integrating above equation, the result is

$$-\frac{1}{xy} + \log x - \log y = \log c$$

$$\Rightarrow -\frac{1}{xy} + \log \frac{x}{y} = \log c$$

$$\Rightarrow \log \frac{x}{y} - \log c = \frac{1}{xy}$$

$$\Rightarrow \log \frac{x}{yc} = \frac{1}{xy} \Rightarrow \frac{x}{yc} = e^{1/xy} \Rightarrow x = cye^{1/xy}$$

Example - 3 :

Find Integrating factor and solve

1. $3y dx + 2x dy = 0$

2. $y \cos x dx + 2 \sin x dy = 0$

3. $2dx + \sec x \cos y dy = 0, \quad y(0) = 0,$

4. $2 \sin y dx + \cos y dy$

Solution. 1. $3y dx + 2x dy = 0$

Multiplying above equation by x^2y we get :

$3x^2y^2 dx + 2x^2y dy = 0$ which is now exact differential equation as

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So x^2y is integrating factor.

The solution is obtained as $d(x^3y^2) = 0$ or $x^3y^2 = c$.

2. $y \cos x dx + 2 \sin x dy = 0$ which is not exact

But by multiplying above equation by y we get :

$y^2 \cos x dx + 2y \sin x dy = 0$ which is now an exact differential equation as

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So y is an integrating factor.

Then we can find solution by writing the equation as

$d(y^2 \sin x) = 0$ or Integrating we get : $y^2 \sin x = c$.

3. $2dx + \sec x \cos y dy = 0$ which is not exact. So by multiplying above equation by $\cos x$ we

$2 \cos x dx + \cos y dy = 0$. Hence $\cos x$ is the integrating factor.

or $d(2 \sin x + \sin y) = 0$, Integrating we get $2 \sin x + \sin y = c$.

But initial value problem is $y(0) = 0$ so $c = 0$.

Then solution is $2 \sin x + \sin y = 0$.

$$12. (x^2y - 2xy^2) dx - (x^3 - 3x^2y) dy = 0.$$

$$\text{Answer : } \frac{x}{y} - 2 \log x + 3 \log y = C$$

$$13. (xy \sin xy + \cos xy) y dx + (xy \sin xy - \cos xy) x dy = 0.$$

$$\text{Answer : } \frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = C$$

$$14. y(1 + xy) dx + x(1 + xy + x^2y^2) dy = 0.$$

$$\text{Answer : } \frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = C$$

$$15. (x - y^2) dx + 2xy dy = 0.$$

$$\text{Answer : } y^2 = cx - x \log x.$$

$$16. (x \sec^2 y - x^2 \cos y) dy = (\tan y - 3x^4) dx.$$

$$\text{Answer : } \frac{-1}{x} \tan y - x^3 + \sin y = C$$

$$17. (2x^4 y^4 e^x + 2xy^3 + y) dx + (x^2 y^4 e^y - x^2 y^2 - 3x) dy = 0.$$

$$\text{Answer : } x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} = C$$

$$18. (y^2 + 2yx^2) dx + (2x^3 - xy) dy = 0.$$

$$\text{Answer : } 4(xy)^{\frac{1}{2}} - \frac{2}{3} \left(\frac{y}{x} \right)^{3/2} = C$$

$$19. x(3y dx + 2x dy) + 8y^4 (y dx + 3x dy) = 0.$$

$$\text{Answer : } x^2 y^2 (x + 4y^4) = C.$$

$$20. y(xy + 2x^2y^2) dx + x(xy - x^2y^2) dy = 0.$$

$$\text{Answer : } 2 \ln x - \ln y - \frac{1}{xy} = C.$$

3.4 LAGRANGE'S METHOD : [Constrained Maximisation]

In many problems, it is necessary to find the maximum or minimum of a function of several variables in which the variables are connected by some given relation or condition which are known as constraints. In such cases Lagrange's method of undetermined multiplier is quite useful to find the solution.

$$\text{Let } u = f(x, y, z)$$

where, x, y, z are related by a known constraint $\phi(x, y, z) = 0$(1)

So, for strationary value of u , $f_x = f_y = f_z = 0$(2)

$$\text{i.e., } \frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0$$

Then the total differential is $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$(3)

Also the total differential equation (2) is given by,

$$d\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = 0 \quad \dots\dots(4)$$

An additional unknown constant λ known as Lagrange's multiplier is introduced so that equation (4) is multiplied by λ we get, $\lambda \frac{\partial\phi}{\partial x} dx + \lambda \frac{\partial\phi}{\partial y} dy + \lambda \frac{\partial\phi}{\partial z} dz = 0$.

Adding the above equation with equation (3) we get,

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial\phi}{\partial x} \right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial\phi}{\partial y} \right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial\phi}{\partial z} \right) dz = 0.$$

As x, y, z are independent variables the above equation will be satisfied if

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial\phi}{\partial x} = 0 \quad \dots\dots(5)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial\phi}{\partial y} = 0 \quad \dots\dots(6)$$

$$\text{and } \frac{\partial f}{\partial z} + \lambda \frac{\partial\phi}{\partial z} = 0. \quad \dots\dots(7)$$

Solving equations (5), (6) and (7) and equation (2) for the four unknowns x, y, z and λ , the stationary points of $f(x, y, z)$ subject to the constraint can be obtained.

Method : (1) Find $F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$

(2) Partially differentiate $F(x, y, z)$ with respect to x, y, z respectively and equate them to

zero, i.e., find $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$.

(3) Solve the four equations $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$ and $\phi(x, y, z) = 0$.

To obtain the values of x, y and z .

Note : This method can be extended to function of several n variables

$$F(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) + \sum_{i=1}^k \lambda_i \phi_i(x_1, x_2, \dots, x_n).$$

Example - 1 : Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$.

Answer : Here $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint equation is $\phi(x, y, z) = xyz - a^3$

\therefore Auxiliary function $F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$

i.e., $F(x, y, z) = x^2 + y^2 + z^2 + \lambda (xyz - a^3)$.

$\dots\dots(1)$

LEARNING OBJECTIVES

- Recapitulation of vectors
- Addition of two vectors
- Parallelogram law of vector addition
- Properties of vector addition
- Rectangular components of a vector (in 3 dimension)
- Position vector
- Scalar product
- Vector or cross product
- Rotation of co-ordinate axes
- Scalar triple product
- Vector triple product

4.1 RECAPITULATION OF VECTORS

A quantity which needs both the magnitude and direction for its complete description is called vector. In addition the quantity should obey vector laws of addition and multiplication.

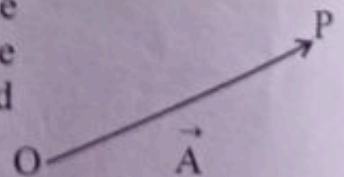
Some examples of vector are displacement, velocity acceleration, Force, Linear momentum, weight, Angular Momentum, Magnetic field, Electric field etc.

- Scalar or scalar quantity is that quantity which has magnitude only and no direction.

Some examples are mass, time, distance, speed, temperature, work, power, energy, volume etc.

Representation of a Vector :

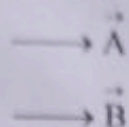
A vector is represented graphically by a straight line of suitable length drawn according to a chosen scale and putting an arrow mark at the end which represents the direction of given vector. The length gives the magnitude of the vector. The initial point of vector is called the origin or tail while the final point is called head or terminus or terminal point.



A vector is represented by placing an arrow on the corresponding letter such as \vec{A} and its magnitude or modulus of a vector \vec{A} is represented by $|\vec{A}|$ or simply A .

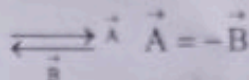
Types of Vectors :

1. **Equal vectors.** Two vectors are said to be equal if they have same magnitude and direction.



$\therefore |\vec{A}| = |\vec{B}|$ and they point in same direction.

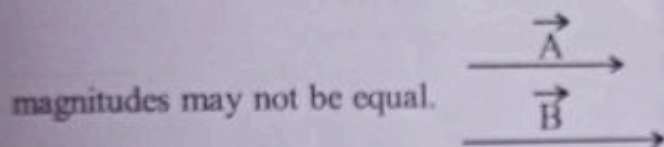
2. **Negative vectors.** Two vectors are said to be negative of one another if their magnitudes are equal and they have opposite direction. $\vec{A} = -\vec{B}$



3. **Null (zero) vector.** A vector having zero magnitude and any arbitrary direction is called Null or Zero vector.

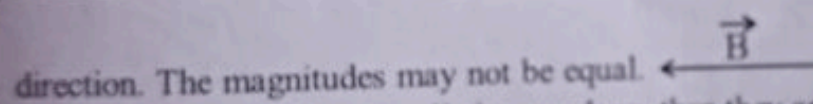
4. **Collinear vectors.** Vectors having same line of action are called collinear vectors.

(a) **Parallel vectors :** Two vectors \vec{A} and \vec{B} are parallel if they point in same direction. The



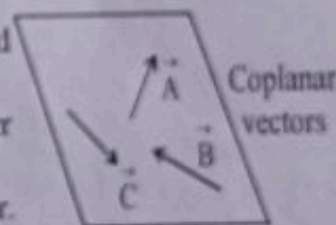
magnitudes may not be equal.

(b) **Parallel vectors :** Two vectors \vec{A} and \vec{B} are said to be anti-parallel if they are in opposite



direction. The magnitudes may not be equal.

5. **Coplanar vectors.** If vectors lie in one plane, then they are said to be coplanar vectors.



6. **Orthogonal vectors.** If vectors are perpendicular to each other then they are called orthogonal vectors.

7. **Unit vectors.** A vector having unit magnitude is called unit vector.

It shows the direction of given vector. When any vector \vec{A} is

divided by its magnitude $|\vec{A}|$, then it gives a unit vector. It is denoted by \hat{A} .

$$\text{So, } \hat{A} = \frac{\vec{A}}{|\vec{A}|} \text{ or } \hat{A} = \frac{\vec{A}}{A}$$

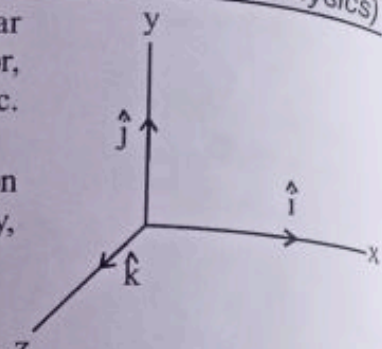


If unit vectors are specified along positive X, Y and Z axes then they are represented by \hat{i}, \hat{j} and \hat{k} .

8. **Like vectors.** The vectors with the same direction are called like vectors.

9. *Polar vectors.* Vectors having a fixed origin and have linear directional effect are called polar vectors position vector, linear velocity, linear momentum, acceleration force etc. are some examples of polar vectors.

10. *Axial vectors.* Vectors directed along the axis of rotation are called Axial vectors. Examples are angular velocity, angular momentum, torque etc.



4.2 ADDITION OF TWO VECTORS

Two vectors \vec{A} and \vec{B} can be added to give the resultant vector \vec{R}

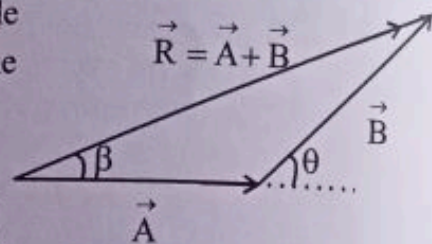
\vec{R} according to either

(i) triangle law of vector addition. or (ii) parallelogram law of vector addition.

Triangle Law :

If \vec{A} and \vec{B} are represented by the two sides of a triangle taken in same order (sense) then the resultant is represented by the third side of triangle taken in opposite order.

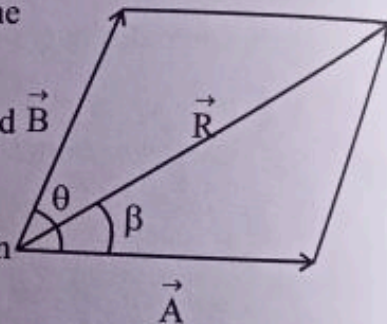
So, $\vec{A} + \vec{B} = \vec{R}$



4.3 PARALLELOGRAM LAW OF VECTOR ADDITION

If the two adjacent sides of a parallelogram represent two vectors then the diagonal of the parallelogram will represent the resultant (vector sum of \vec{A} and \vec{B}).

From both the laws of vector addition we get magnitude and direction of resultant of two vectors $\vec{A} + \vec{B}$.



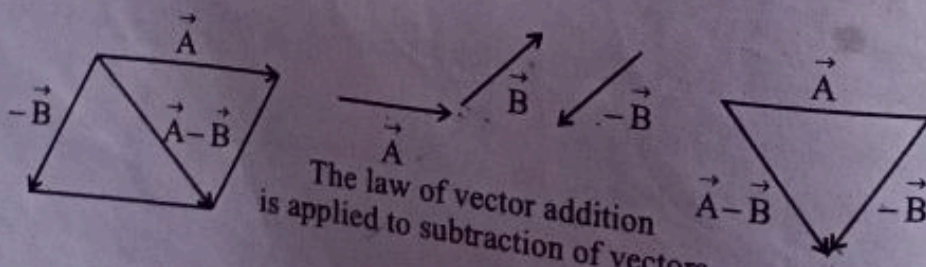
So, $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$ where θ is angle between \vec{A} and \vec{B} .

$\tan\beta = \frac{B\sin\theta}{A + B\cos\theta} \Rightarrow \beta = \tan^{-1}\left(\frac{B\sin\theta}{A + B\cos\theta}\right)$

where, β is the angle between \vec{R} and \vec{A} .

Subtraction of Two Vectors :

The subtraction of vector \vec{B} from vector \vec{A} is same as the addition of vector $-\vec{B}$ to vector \vec{A} . Hence $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$.



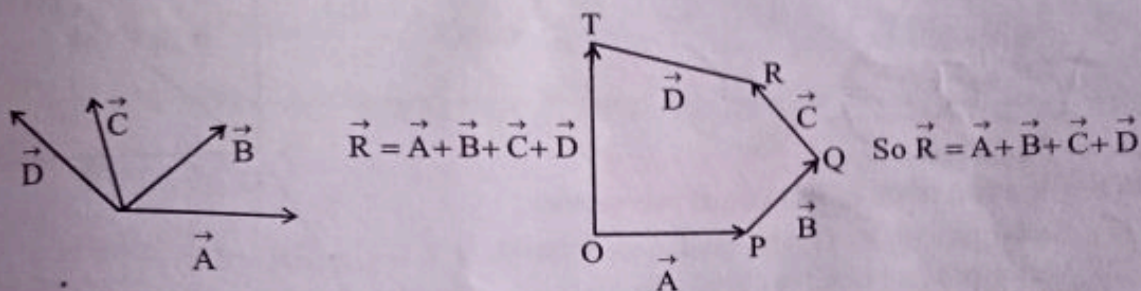
4.4 PROPERTIES OF VECTOR ADDITION

(i) It obeys commutative law, i.e., $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

(ii) It obeys associative law, i.e., $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C} = \vec{A} + \vec{B} + \vec{C}$.

Polygon Law of Vector Addition :

The addition or sum of more than two vectors can be obtained using polygon law of vector addition. According to this law if a no. of vectors acting simultaneously at a point are represented in magnitude and direction by the sides of an open polygon taken in same order then the resultant is represented by the closing side of the polygon taken in opposite order.



4.5 RECTANGULAR COMPONENTS OF A VECTOR (IN 3 DIMENSION)

Let \vec{A} make angles, α , β and γ with x, y and z axes respectively.

Let \hat{i} , \hat{j} and \hat{k} be unit vectors along x, y and z axes respectively.

Let $\vec{OP} = \vec{A}$, $\vec{ON} = \vec{A}_x$, $\vec{OT} = \vec{A}_y$, $\vec{OS} = \vec{A}_z$

So $\vec{OM} + \vec{MP} = \vec{OP}$ or $\vec{OM} + \vec{OT} = \vec{OP}$ (1)

Also $\vec{OS} + \vec{SM} = \vec{OM}$ or $\vec{OS} + \vec{ON} = \vec{OM}$ (2)

Then equation (1) becomes $\vec{OP} = \vec{OS} + \vec{ON} + \vec{OT}$

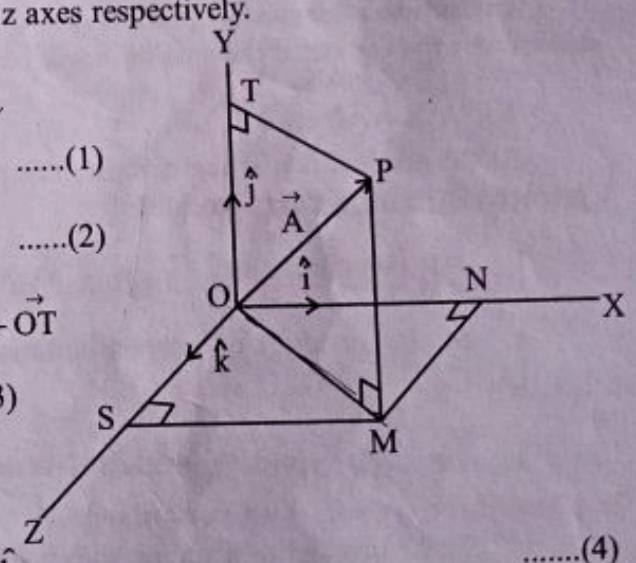
or $\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$ (3)

But $\vec{A}_x = A_x \hat{i}$, $\vec{A}_y = A_y \hat{j}$, $\vec{A}_z = A_z \hat{k}$

Equation (3) becomes $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ (4)

\vec{A}_x , \vec{A}_y and \vec{A}_z are Rectangular (vector) components of \vec{A} in cartesian coordinate system.

The magnitude of \vec{A} is given by, $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ (5)



The quantities $\cos\alpha = \frac{A_x}{A} = \ell$, $\cos\beta = \frac{A_y}{A} = m$, $\cos\gamma = \frac{A_z}{A} = n$ are known as direction

cosines of \vec{A} . So $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{A_x^2}{A^2} + \frac{A_y^2}{A^2} + \frac{A_z^2}{A^2}$.

or $\ell^2 + m^2 + n^2 = 1$

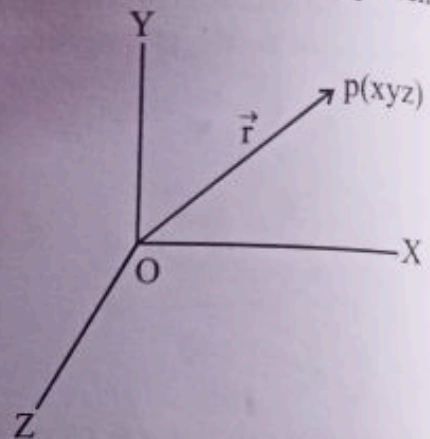
4.6 POSITION VECTOR

It is the vector joining the point P in space to the origin of co-ordinate system. It is represented

by $\vec{OP} = \vec{r}$.

So, $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$r = \sqrt{x^2 + y^2 + z^2}$



Product of Two Vector :

There are two types of product of two vectors :

- (1) Scalar product or Dot product.
- (2) Vector product or Cross product.

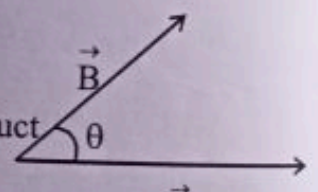
4.7 SCALAR PRODUCT

The scalar product of two vectors \vec{A} and \vec{B} is denoted by $\vec{A} \cdot \vec{B}$ (read as \vec{A} dot \vec{B}). Hence it is called dot product of two vectors.

Definition : The scalar product of two vectors is defined as product of magnitude of two vectors and the cosine of the smaller angle between the two vectors.

So, $\vec{A} \cdot \vec{B} = AB \cos\theta$

As the product is a scalar quantity this product is called scalar product



Characteristics of Scalar Product :

1. It is commutative $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ as $\vec{A} \cdot \vec{B} = AB \cos\theta$ and $\vec{B} \cdot \vec{A} = BA \cos(-\theta) = AB \cos\theta$.
2. Scalar product of perpendicular vectors. If \vec{A} and \vec{B} are perpendicular to each other, $\theta = 90$.

So, $\vec{A} \cdot \vec{B} = AB \cos 90 = 0$, which is condition for Two vectors \vec{A} and \vec{B} to be mutually perpendiculars, ($A \neq 0, B \neq 0$) or orthogonal.

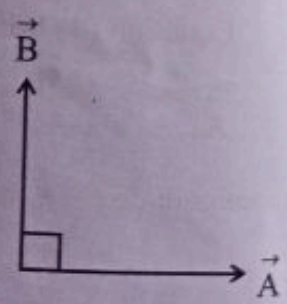
3. Scalar product of a vector with itself.

$\vec{A} \cdot \vec{A} = A A \cos 0 = A^2$

4. Scalar product of mutually perpendicular unit vectors,

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

But $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$.



5. Scalar product of collinear vectors :

(a) For parallel vectors $\theta = 0^\circ$.

$$\text{So, } \vec{A} \cdot \vec{A} = AB \cos 0 = AB$$

(b) For Anti-parallel vectors $\theta = 180^\circ$.

$$\vec{A} \cdot \vec{B} = AB \cos 180 = -AB.$$

6. Geometrical meaning of scalar product :

Scalar product of two vectors is the product of the magnitude of either vector and the projection or component of the other in its direction.

$$\text{In } \Delta OQM, \cos \theta = \frac{OM}{OQ} \text{ or } OQ \cos \theta = OM$$

or $OM = B \cos \theta$ is the projection or component of

\vec{B} along \vec{A} .

Hence $\vec{A} \cdot \vec{B} = AB \cos \theta = A$ (Projection or component of \vec{B} along \vec{A}).

Similarly $\vec{B} \cdot \vec{A} = BA \cos \theta = B$ (Projection or component of \vec{A} along \vec{B}).

7. Product of scalar product with a vector :

The scalar product is a number. So it can occur as a numerical co-efficient of a vector.

E.g., $(\vec{A} \cdot \vec{B}) \vec{C}$ is a vector of magnitude $(\vec{A} \cdot \vec{B}) \vec{C}$ in the direction of \vec{C} .

8. Scalar product of Two vectors is distributive :

$$\text{i.e., } \vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

$(\vec{B} + \vec{C})$ is the resultant of \vec{B} and \vec{C} .

So, the scalar product of \vec{A} and $(\vec{B} + \vec{C}) = \vec{D}$ (let) is given by $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{D} = A$.

(Projection of \vec{D} along \vec{A})

or $\vec{A} \cdot (\vec{B} + \vec{C}) = A(ON)$ (\because ON is projection or component of $\vec{D} = (\vec{B} + \vec{C})$ along \vec{A}).

or $\vec{A} \cdot (\vec{B} + \vec{C}) = A(OM + MN)$

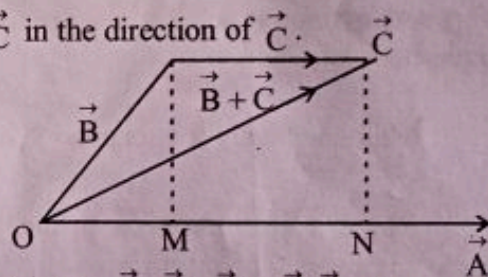
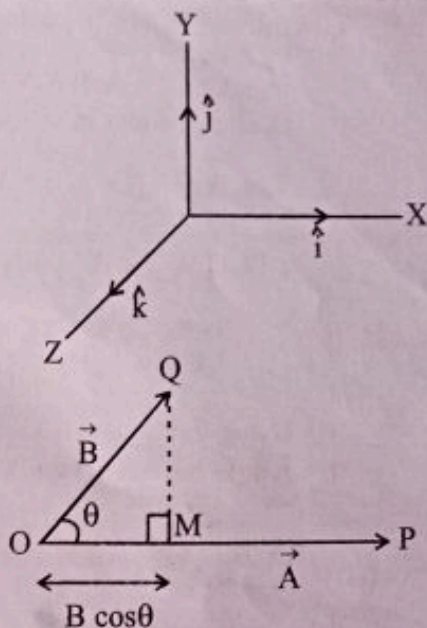
or $\vec{A} \cdot (\vec{B} + \vec{C}) = A(OM) + A(MN)$

....(1)

But OM is projection of \vec{B} along \vec{A}

MN is projection of \vec{C} along \vec{A}

Hence $A(OM) = \vec{A} \cdot \vec{B}$ and $A(MN) = \vec{A} \cdot \vec{C}$.



Using these values in equation (1) we get :

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

9. Scalar product in terms of components.

$$\text{Let } \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}, \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\therefore \vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$\boxed{\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z}$$

Using the fact $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$, $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$.

10. Angle between Two Vectors :

$$\vec{A} \cdot \vec{B} = AB \cos \theta \Rightarrow \cos \theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2} \sqrt{B_x^2 + B_y^2 + B_z^2}}$$

$$\text{or } \theta = \cos^{-1} \left\{ \frac{(A_x B_x + A_y B_y + A_z B_z)}{\sqrt{A_x^2 + A_y^2 + A_z^2} \sqrt{B_x^2 + B_y^2 + B_z^2}} \right\}$$

4.8 VECTOR OR CROSS PRODUCT

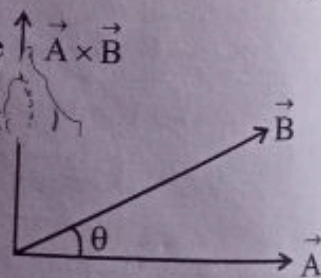
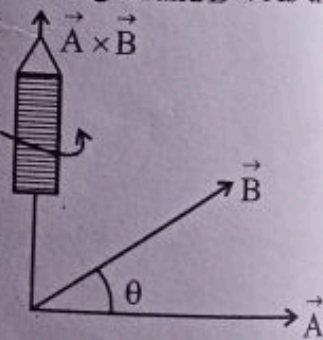
Definition : The vector or cross product of two vectors \vec{A} and \vec{B} is defined as a vector whose magnitude is equal to the product of magnitudes of two vectors and the sine of the smaller angle between them.

Mathematically it is expressed as, $\boxed{\vec{A} \times \vec{B} = (AB \sin \theta) \hat{n}}$.

where \hat{n} is a unit vector which is perpendicular to the plane containing \vec{A} and \vec{B} . As the product is a vector quantity, cross-product of two vectors is called vector product.

The direction of \hat{n} is obtained using right hand rule according to which if a right-handed screw is rotated in direction \vec{A} to \vec{B} through the smaller angle then the direction in which tip of the screw advances given the direction of $(\vec{A} \times \vec{B})$ or \hat{n} .

Otherwise it can be stated as if the fingers of the right hand are turned in the direction from \vec{A} towards \vec{B} then the thumb points along the direction of \hat{n} or $\vec{A} \times \vec{B}$.



Characteristics of Vector Product :

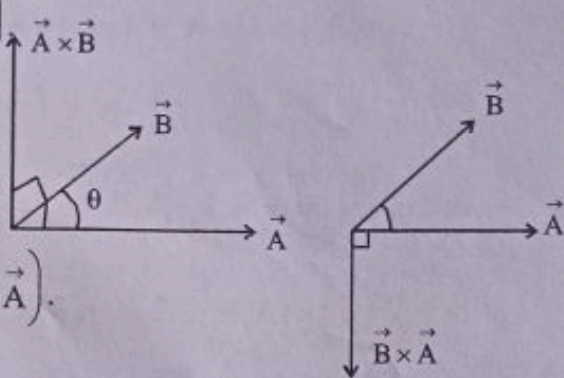
1. It is not commutative

i.e., $(\vec{A} \times \vec{B}) \neq (\vec{B} \times \vec{A})$ but $(\vec{A} \times \vec{B}) = -(\vec{B} \times \vec{A})$

as $\vec{A} \times \vec{B} = (AB \sin \theta) \hat{n}$

$\vec{B} \times \vec{A} = BA \sin(-\theta) \hat{n} = -AB \sin \theta \hat{n}$

$\Rightarrow (\vec{A} \times \vec{B}) = -(\vec{B} \times \vec{A})$ or $(\vec{A} \times \vec{B}) \neq (\vec{B} \times \vec{A})$.



2. Vector product of A vector with itself.

$\vec{A} \times \vec{A} = AA \sin 0 \hat{n} = \text{Null vector.}$

3. Vector product of perpendicular vectors when

$\theta = \frac{\pi}{2}$ then $\vec{A} \times \vec{B} = \left(AB \sin \frac{\pi}{2} \right) \hat{n} = AB \hat{n}.$

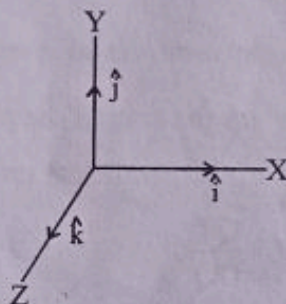
4. Vector product of collinear vectors.

(a) For parallel vectors $\theta = 0^\circ$.

$\vec{A} \times \vec{B} = AB \sin 0 \hat{n} = \text{Null vector.}$

(b) For Anti-parallel vectors $\theta = 180^\circ$.

$\vec{A} \times \vec{B} = AB \sin 180 \hat{n} = \text{Null vector.}$



5. Vector product of unit vectors.

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \text{Null vector.}$

$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$ but $\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$.

6. Unit vector perpendicular to both the vectors, we know $(\vec{A} \times \vec{B}) = (AB \sin \theta) \hat{n}$.

or $\hat{n} = \frac{\vec{A} \times \vec{B}}{AB \sin \theta} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$.

7. Vector product in terms of rectangular components.

Let $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z, \vec{B} = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z$

$\therefore (\vec{A} \times \vec{B}) = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) \times (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) = \hat{i}A_x \times (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z)$

$$\begin{aligned}
 & + \hat{j}A_y \times (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) + \hat{k}A_z \times (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) \\
 & = A_x B_x (\hat{i} \times \hat{i}) + A_x B_y (\hat{i} \times \hat{j}) + A_x B_z (\hat{i} \times \hat{k}) + A_y B_x (\hat{j} \times \hat{i}) + A_y B_y (\hat{j} \times \hat{j}) \\
 & \quad + A_y B_z (\hat{j} \times \hat{k}) + A_z B_x (\hat{k} \times \hat{i}) + A_z B_y (\hat{k} \times \hat{j}) + A_z B_z (\hat{k} \times \hat{k}). \\
 \text{or } (\vec{A} \times \vec{B}) & = A_x B_y \hat{k} - A_x B_z \hat{j} - A_y B_x \hat{k} + A_y B_z \hat{i} + A_z B_x \hat{j} - A_z B_y \hat{i} \\
 \text{or } (\vec{A} \times \vec{B}) & = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k} \\
 \text{or } (\vec{A} \times \vec{B}) & = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}
 \end{aligned}$$

Geometrical Meaning :

8. If \vec{A} and \vec{B} represent the adjacent sides of a parallelogram inclined to each other at an angle θ then $|\vec{A} \times \vec{B}|$ gives the area of the parallelogram.

QT is the height of the parallelogram.

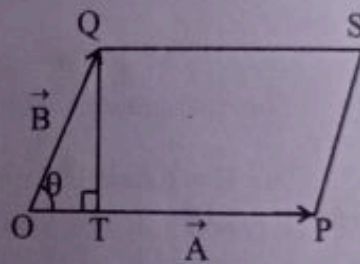
In $\Delta OQT, \sin \theta = \frac{QT}{OQ}$

or $QT = OQ \sin \theta$ or $QT = B \sin \theta$ as $OQ = B$.

\therefore Area of parallelogram OPSQ = (Base) (height)

$= (OP) (QT) = AB \sin \theta = |\vec{A} \times \vec{B}|$.

$\Rightarrow |\vec{A} \times \vec{B}|$ gives the area of the parallelogram.



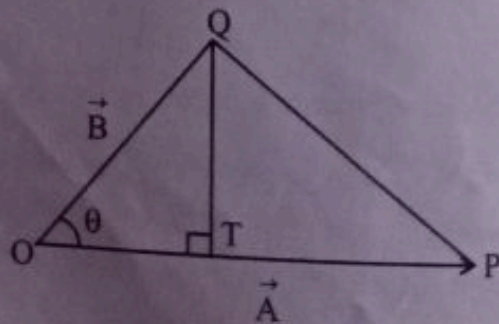
9. If \vec{A} and \vec{B} represent the adjacent sides of a triangle inclined to each other at an angle θ then $\frac{1}{2} |\vec{A} \times \vec{B}|$ represents area of the triangle.

QT is the height of triangle.

In $\Delta OQT, \sin \theta = \frac{QT}{OQ}$

$\Rightarrow QT = OQ \sin \theta = B \sin \theta$ as $OQ = B$.

Area of triangle OPQ = $\frac{1}{2}$ (Base) (Height)



$$= \frac{1}{2}(\text{OP})(\text{QT}) = \frac{1}{2}AB \sin \theta = \frac{1}{2}|\vec{A} \times \vec{B}|$$

i.e., $\frac{1}{2}|\vec{A} \times \vec{B}|$ gives Area of the Triangle.

10. Vector product of Two vectors is distributive i.e., $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$.

$$\text{Let } \vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z, \vec{B} = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z, \vec{C} = \hat{i}C_x + \hat{j}C_y + \hat{k}C_z$$

$$\therefore \vec{B} + \vec{C} = \hat{i}(B_x + C_x) + \hat{j}(B_y + C_y) + \hat{k}(B_z + C_z)$$

$$\vec{A} \times (\vec{B} + \vec{C}) = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) \times [\hat{i}(B_x + C_x) + \hat{j}(B_y + C_y) + \hat{k}(B_z + C_z)]$$

$$= A_x(B_y + C_y)\hat{k} - \hat{j}A_x(B_z + C_z) - \hat{k}A_y(B_x + C_x)$$

$$+ \hat{i}A_y(B_z + C_z) + \hat{j}A_z(B_x + C_x) - \hat{i}A_z(B_y + C_y)$$

$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

$$+ (A_yC_z - A_zC_y)\hat{i} + (A_zC_x - A_xC_z)\hat{j} + (A_xC_y - A_yC_x)\hat{k}$$

$$\text{or } \vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

Some Examples of Product of Two Vectors in Physics :

1. Work done As Scalar Product :

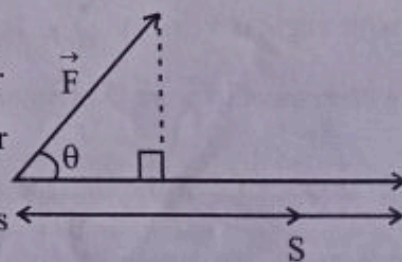
If a Force \vec{F} (assumed to be constant) acts on a body such that \vec{S} is the displacement and θ is the angle between \vec{F} and \vec{S} then work is defined as the product of the component of force along direction of displacement and magnitude of displacement.

So work $W = (F \cos \theta) S = \vec{F} \cdot \vec{S}$ which is a scalar quantity.

If various forces, $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n$ act on a particle for

displacement \vec{r} then Total work done is

$W = \sum \vec{F}_i \cdot \vec{r} = \vec{r} \cdot \sum \vec{F}_i = \vec{r} \cdot \vec{F}$ where $\vec{F} = \sum \vec{F}_i$ is the vector sum of all the forces acting on the particle.



2. Instantaneous Power :

It is given by $P = \frac{dw}{dt}$, but $W = \vec{F} \cdot \vec{S}$.

So $P = \frac{d}{dt}(\vec{F} \cdot \vec{S}) = \vec{F} \cdot \frac{d\vec{S}}{dt} = \vec{F} \cdot \vec{V}$ as \vec{V} is velocity of the body.

3. Flux of a Vector :

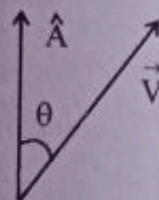
If \vec{V} is a vector inclined at angle θ with the outward unit normal \hat{A} , then flux of vector = (Normal component of vector) (Area).

or Flux of vector = $(V \cos \theta)A = \vec{V} \cdot \vec{A}$.

On the same procedure we can have

Electric flux $\phi_E = \vec{E} \cdot \vec{A}$, \vec{E} = Electric field.

Magnetic flux $\phi_B = \vec{B} \cdot \vec{A}$, \vec{B} = Magnetic field.



4. Angular Momentum :

The angular momentum of a particle having linear momentum

$\vec{P} = m\vec{V}$ is given by $\vec{L} = (\vec{r} \times \vec{p})$.

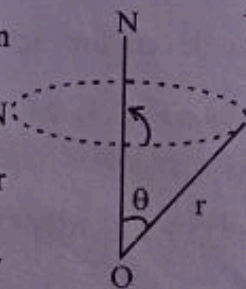
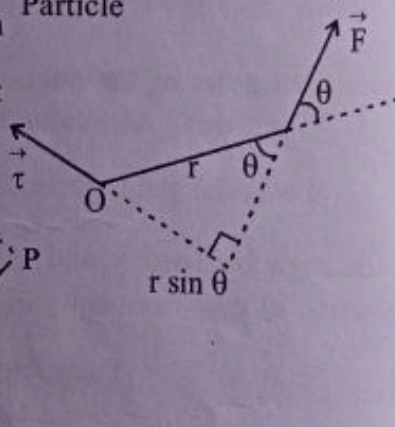
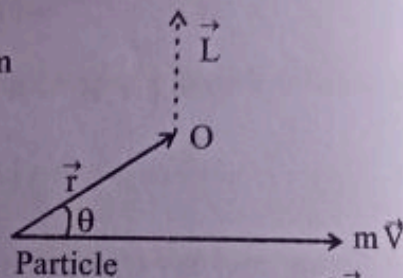
\vec{r} is the position vector of particle from a point.

5. Torque : Torque or moment of a force about a point is given

by $\vec{\tau} = \vec{r} \times \vec{F}$ where \vec{r} is position vector with respect to point 0.

6. Angular velocity of a rigid body about an Axis.

If a rigid body rotates about an axis ON: with angular velocity $\vec{\omega}$, \vec{r} is the position vector with respect to point O of point P, then $\vec{v} = \vec{\omega} \times \vec{r}$.



Example - 1 : Show that $\vec{A} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{B} = 5\hat{i} + 2\hat{j} + 4\hat{k}$ are perpendicular to each other.

Answer : For perpendicular vectors $\vec{A} \cdot \vec{B} = 0$.

So, $\vec{A} \cdot \vec{B} = (2\hat{i} + 3\hat{j} - 4\hat{k}) \cdot (5\hat{i} + 2\hat{j} + 4\hat{k})$ $\left\{ \because \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \right\}$
 $= (2)(5) + (3)(2) + (-4)(4) = 10 + 6 - 16 = 0$.

Hence \vec{A} and \vec{B} are perpendicular to each other.

Example - 15 : If a force $\vec{F} = -3\hat{i} + 2\hat{j} + 2\hat{k}$ acts as a point $2\hat{i} + 3\hat{j} + 4\hat{k}$, find the moment of force about the origin.

Answer : Moment of force is $(\vec{r} \times \vec{F}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times (-3\hat{i} + 2\hat{j} + 2\hat{k}) = -2\hat{i} - 16\hat{j} + 13\hat{k}$.

Example - 16 : A rigid body is rotating with angular velocity 5 radian per second about an axis parallel to $4\hat{i} - 2\hat{j} + \hat{k}$ through the point $2\hat{i} - \hat{j} + \hat{k}$. Find the magnitude of the velocity of point of rigid body at $3\hat{i} + 2\hat{j} - \hat{k}$.

Answer : The position vector is given by

$$\vec{r} = (3\hat{i} + 2\hat{j} - \hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = \hat{i} + 3\hat{j} - 2\hat{k}$$

$$\text{Angular velocity } \vec{\omega} = \omega \hat{n} = 5 \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{4^2 + (-2)^2 + (1)^2}}$$

$$\text{or } \vec{\omega} = \frac{5(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$$

So, velocity of the point $\vec{v} = \vec{\omega} \times \vec{r}$.

$$= \frac{5}{\sqrt{21}} (4\hat{i} - 2\hat{j} + \hat{k}) \times (\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{5}{\sqrt{21}} (\hat{i} + 9\hat{j} + 14\hat{k})$$

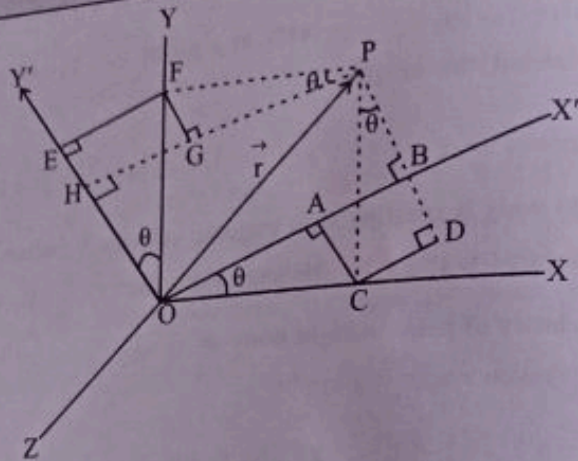
$$\text{Its magnitude is given by, } v = \frac{5}{\sqrt{21}} \sqrt{1 + 81 + 196} = \frac{5}{\sqrt{21}} \sqrt{278} = 18.2.$$

4.9 ROTATION OF CO-ORDINATE AXES : (PROPERTIES OF VECTOR UNDER ROTATION)

Let us discuss the manner in which a vector transforms under rotation. Let us consider the three dimensional cartesian coordinate system (x, y, z). Let the rotation be around z axis i.e x, y co-ordinates are rotated counter clockwise through angle θ keeping position vector \vec{r} fixed. Let

$\vec{OP} = \vec{r}$ where P is the point with the coordinates x, y and z, such that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Similarly let x' , y' and z' be the co-ordinates of the point when axis is rotated by an angle θ in the new coordinate system (primed system). It is shown in the figure below.



In ΔOAC , $\cos \theta = \frac{OA}{OC}$ or $OA = OC \cos \theta$ $OA = x \cos \theta$

In ΔCDP , $\sin \theta = \frac{CD}{PC}$ or $CD = PC \sin \theta$ $CD = y \sin \theta$ But $AB = CD$.

Hence $OB = OA + AB$ i.e., $OB = x \cos \theta + y \sin \theta$.

or $x' = x \cos \theta + y \sin \theta$ (1)

as $OB = x'$ along X' axis.

Now in ΔOEF , $\cos \theta = \frac{OE}{OF}$ or $OE = OF \cos \theta$ or $OE = y \cos \theta$ as $OF = y$.

Also in ΔFPG , $\sin \theta = \frac{FG}{PF}$ or $FG = PF \sin \theta$

or $FG = x \sin \theta$. But $FG = EH$.

Then $OH = OE - EH$ or $y' = y \cos \theta - x \sin \theta$.

or $y' = -x \sin \theta + y \cos \theta$ (2) as $OH = y'$ along Y' axis.

Finally $z' = z$ (3)

Hence equation (1), (2) and (3) can be written as

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or $\vec{r}' = R(\theta) \vec{r}$

.....(4) using $\vec{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ and $r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

\vec{r} to \vec{r}' under rotation
So, $R(\theta)$ is called rotation matrix
Also we know that the components of a vector relative to the vector components

axes. So let (A_x', A_y') be the components of \vec{r}' along X' and Y' axes. (1) and (2) we have

$$A_x' = A_x \cos \theta + A_y \sin \theta$$

$$A_y' = -A_x \sin \theta + A_y \cos \theta$$

Component of \vec{r}' along Z' axis i.e., $A_z' = A_z$

In terms of x, y, z

Also under rotation

that the magnitude of the vector

$$= \sqrt{(A_x')^2 + (A_y')^2 + (A_z')^2}$$

$$= \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Hence though the magnitude of the vector remains invariant under rotation of the coordinate axes. This is an important property of vectors.

vector \vec{A} remains invariant

components of vector \vec{A} under rotation of the coordinate system. (The coordinate system is rotated through an angle θ about the Z axis.)

Invariance of Scalars

Scalars are invariant under rotation of the coordinate system.

product of Two Vectors

If we consider two vectors \vec{a} and \vec{b} in the xy plane as discussed in previous chapter, their components along x, y axes through

and $R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which changes the position vector.

\vec{r} to \vec{r}' under rotation of coordinate axes about z-axis in counterclockwise sense.

So, $R(\theta)$ is called Transformation matrix or rotation matrix.

Also we know that vector can be represented by coordinates of a point which are proportional to the vector components. So the components of vector should transform under rotation of coordinate axes. So let (A_x', A_y') be the components in primed coordinate system under rotation and (A_x, A_y) be the components of vector in the unprimed coordinate system, then following equation (1) and (2) we have,

$$A_x' = A_x \cos \theta + A_y \sin \theta. \quad \dots(5)$$

$$A_y' = -A_x \sin \theta + A_y \cos \theta. \quad \dots(6)$$

Component of A along Z axis remains unchanged.

$$\text{i.e., } A_z' = A_z.$$

In terms of x and y components we find the magnitude of \vec{A} as $A = \sqrt{A_x^2 + A_y^2} \quad \dots(7)$

Also under rotation of coordinate system the new components of vector are A_x' and A_y' so that the magnitude of the new vector under rotation about z-axis will be given by $\sqrt{A_x'^2 + A_y'^2}$.

$$\begin{aligned} &= \sqrt{(A_x \cos \theta + A_y \sin \theta)^2 + (-A_x \sin \theta + A_y \cos \theta)^2} \\ &= \sqrt{A_x^2 + A_y^2} \text{ which is same as in equation (7).} \end{aligned}$$

Hence though the components of a vector may change under rotation by some angle, but the magnitude of the vector in the unprimed and primed coordinate system remains invariant. (The magnitude of the vector is a scalar quantity and so it must be invariant under rotation of coordinate axes. This is an important property of vector. Also it can be shown that the direction of the given vector \vec{A} remains unchanged or invariant under rotation of the coordinate axes though the

components of vector may change to new values. Hence the vector \vec{A} is independent of the rotation of the coordinate system. The vector equations are independent of any particular coordinate system. (The coordinate system may be cartesian or any other system).

Invariance of Scalar product of Two Vector under Rotation :

Scalars are independent of the coordinate system. Let us examine the behaviour of scalar product of Two Vectors $\vec{A} \cdot \vec{B}$ under rotation of the coordinate axes.

If we consider the rotation of the cartesian coordinate system about Z-axis by angle θ then as discussed in previous article, the transformation of the components of vector \vec{A} under rotation of x y axes through angle θ about Z-axis are given as

$$\begin{aligned} A'_x &= A_x \cos \theta + A_y \sin \theta \\ A'_y &= -A_x \sin \theta + A_y \cos \theta \\ A'_z &= A_z \end{aligned}$$

Similarly the components of vector \vec{B} under rotation are represented as

$$\begin{aligned} B'_x &= B_x \cos \theta + B_y \sin \theta \\ B'_y &= -B_x \sin \theta + B_y \cos \theta \\ B'_z &= B_z \end{aligned}$$

Hence as A_x, A_y and A_z are components of \vec{A} along x, y, z axes then

$\vec{A} \cdot \vec{B} = (A_x B_x + A_y B_y + A_z B_z)$ as obtained earlier in the discussion of scalar product of Two Vectors in terms of its components.

So, let us find scalar product of \vec{A} and \vec{B} in terms of new components (A'_x, A'_y, A'_z) and (B'_x, B'_y, B'_z) under rotation.

The result is $A'_x B'_x + A'_y B'_y + A'_z B'_z$.

$$\begin{aligned} &= (A_x \cos \theta + A_y \sin \theta) (B_x \cos \theta + B_y \sin \theta) \\ &\quad + (-A_x \sin \theta + A_y \cos \theta) (-B_x \sin \theta + B_y \cos \theta) + A_z B_z \end{aligned}$$

$$\begin{aligned} \text{or } A'_x B'_x + A'_y B'_y + A'_z B'_z &= A_x B_x \cos^2 \theta + A_x B_y \cos \theta \sin \theta + A_y B_x \sin \theta \cos \theta + A_y B_y \sin^2 \theta \\ &\quad + A_x B_x \sin^2 \theta - A_x B_y \sin \theta \cos \theta - A_y B_x \cos \theta \sin \theta + A_y B_y \cos^2 \theta + A_z B_z \\ &= A_x B_x (\cos^2 \theta + \sin^2 \theta) + A_y B_y (\cos^2 \theta + \sin^2 \theta) + A_z B_z \end{aligned}$$

$$\text{or } \boxed{A'_x B'_x + A'_y B'_y + A'_z B'_z = A_x B_x + A_y B_y + A_z B_z}$$

So, scalar product of Two Vectors is invariant under rotation of coordinate axes.

4.10 SCALAR TRIPLE PRODUCT :

The scalar triple product of three vectors is the scalar product of a vector with vector product of the other two vectors. If, \vec{A}, \vec{B} and \vec{C} be any three vectors then examples of scalar triple product are

$$\vec{A} \cdot (\vec{B} \times \vec{C}), (\vec{A} \times \vec{B}) \cdot \vec{C}, (\vec{B} \times \vec{C}) \cdot \vec{A}, \vec{C} \cdot (\vec{A} \times \vec{B}) \text{ etc.}$$

It is also represented as $\left[\begin{matrix} \vec{A} & \vec{B} & \vec{C} \end{matrix} \right]$ which is a scalar quantity.

Geometrical Interpretation :

The scalar triple product $\vec{A} \cdot (\vec{B} \times \vec{C})$ represents the volume of a parallelopiped having \vec{A}, \vec{B} and \vec{C} as its coterminus edges.

Figure below shows a parallelopiped with $\vec{OP} = \vec{A}, \vec{OQ} = \vec{B}, \vec{OS} = \vec{C}$, as coterminus edges. The altitude of the parallelopiped is $h = A \cos \phi$.

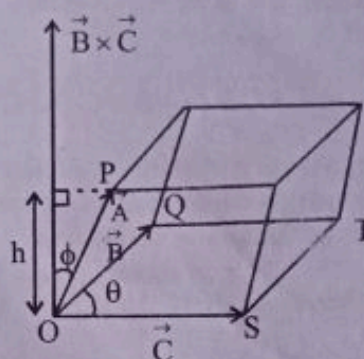
The volume of the parallelepiped.
 = (Area of its base) (vertical height or altitude).
 = (Area of parallelogram OSTQ) h.

$$= |\vec{B} \times \vec{C}| (A \cos \phi)$$

$$= AD \cos \phi \text{ where } \vec{D} = (\vec{B} \times \vec{C}) \text{ (let)}$$

$$= \vec{A} \cdot \vec{D} \quad \therefore D = |\vec{B} \times \vec{C}|$$

$$= \vec{A} \cdot (\vec{B} \times \vec{C})$$



This value of scalar triple product is positive if ϕ is an acute angle. (If $\vec{A}, \vec{B}, \vec{C}$ form a right handed system of vectors.)

Also this value is negative if ϕ is obtuse i.e., $\vec{A}, \vec{B}, \vec{C}$ form a left handed system.

Characteristics of Scalar Triple Product :

1. Condition for three vectors to be coplanar :

If \vec{A}, \vec{B} and \vec{C} are coplanar, then value of scalar triple product is zero. In this case volume of the parallelepiped becomes zero. i.e., $[\vec{A} \vec{B} \vec{C}] = 0$.

2. If any Two vectors of a scalar triple product are equal then the product is zero.

$$\text{i.e., } [\vec{A}, \vec{A}, \vec{B}] = [\vec{A}, \vec{B}, \vec{B}] = [\vec{A}, \vec{C}, \vec{C}] \text{ etc. vanish.}$$

$$\text{As } [\vec{A}, \vec{B}, \vec{B}] = \vec{A} \cdot (\vec{B} \times \vec{B}) = \text{zero.}$$

3. If two of the vectors are parallel the value of scalar triple product is zero. So if \vec{B} is parallel

$$\text{to } \vec{A} \text{ i.e., } \vec{B} = K\vec{A} \text{ where } K \text{ is a scalar then } [\vec{A} \vec{B} \vec{C}] = (\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{A} \times K\vec{A}) \cdot \vec{C} = 0$$

$$\text{as } \vec{A} \times \vec{A} = 0.$$

4. The value of scalar triple product depends on the cyclic order of vectors.

Any one face of the parallelepiped may be taken as its base so volume of parallelepiped may also be given as $\vec{B} \cdot (\vec{C} \times \vec{A})$ and $\vec{C} \cdot (\vec{A} \times \vec{B})$ i.e., cyclic order of \vec{A}, \vec{B} and \vec{C} are maintained.

$$\text{i.e., } [\vec{A} \vec{B} \vec{C}] = [\vec{B} \vec{C} \vec{A}] = [\vec{C} \vec{A} \vec{B}] = V \text{ the volume of parallelepiped.}$$

$$\text{But } \vec{A} \cdot (\vec{C} \times \vec{B}) = \vec{A} \cdot \left\{ -(\vec{B} \times \vec{C}) \right\} = -\vec{A} \cdot (\vec{B} \times \vec{C})$$

$$\text{Hence } [\vec{A} \vec{C} \vec{B}] = [\vec{B} \vec{A} \vec{C}] = [\vec{C} \vec{B} \vec{A}] = -V.$$

5. The value of scalar triple product is independent of the position of dot and cross. In other words the position of dot and cross may be interchanged without changing the value if the cyclic order of vectors is maintained.

$$\text{We have } \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\text{i.e., } \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\text{or } \vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

Hence the position of dot and cross can be interchanged in a scalar triple product.

6. Scalar triple product in terms of components.

$$\text{Let } \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}, \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}, \vec{C} = C_x \hat{i} + C_y \hat{j} + C_z \hat{k}$$

$$\text{Then } (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (B_y C_z - B_z C_y) \hat{i} + (B_z C_x - B_x C_z) \hat{j} + (B_x C_y - B_y C_x) \hat{k}$$

$$\text{Then } \vec{A} \cdot (\vec{B} \times \vec{C}) = (\hat{i} A_x + \hat{j} A_y + \hat{k} A_z) \cdot \left[\hat{i} (B_y C_z - B_z C_y) + \hat{j} (B_z C_x - B_x C_z) + \hat{k} (B_x C_y - B_y C_x) \right]$$

$$\text{or } \vec{A} \cdot (\vec{B} \times \vec{C}) = A_x (B_y C_z - B_z C_y) + A_y (B_z C_x - B_x C_z) + A_z (B_x C_y - B_y C_x)$$

$$\text{or } \vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

Note : If \vec{A}, \vec{B} and \vec{C} lie in one plane i.e., they are coplanar then we know $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$.

$$\text{or } \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = 0 \text{ which is condition for the three vectors } \vec{A}, \vec{B} \text{ and } \vec{C} \text{ to be coplanar.}$$

7. Distributive law holds for scalar triple product,

$$\text{i.e., } \left[\vec{A}, \vec{B} + \vec{D}, \vec{C} + \vec{E} \right] = \left[\vec{A} \vec{B} \vec{C} \right] + \left[\vec{A} \vec{B} \vec{E} \right] + \left[\vec{A} \vec{D} \vec{C} \right] + \left[\vec{A} \vec{D} \vec{E} \right] \text{ because both the scalar and vector product of two vectors are distributive.}$$

4.11 VECTOR TRIPLE PRODUCT

The vector triple product of three vectors is vector product of any vector with vector product of remaining two vectors. Examples of vector triple product are $\vec{A} \times (\vec{B} \times \vec{C}), \vec{B} \times (\vec{C} \times \vec{A}), (\vec{A} \times \vec{B}) \times \vec{C}$ etc. The product is a vector quantity.

$$\text{Expansion Formula : } \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$$

$$\text{Proof : } \vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z, \vec{B} = \hat{i}B_x + \hat{j}B_y + \hat{k}B_z, \vec{C} = \hat{i}C_x + \hat{j}C_y + \hat{k}C_z$$

$$\vec{B} \times \vec{C} = (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) \times (\hat{i}C_x + \hat{j}C_y + \hat{k}C_z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \hat{i}(B_y C_z - B_z C_y) - \hat{j}(B_x C_z - B_z C_x) + \hat{k}(B_x C_y - B_y C_x)$$

$$\therefore \vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ (B_y C_z - B_z C_y) & (B_z C_x - B_x C_z) & (B_x C_y - B_y C_x) \end{vmatrix}$$

$$\Rightarrow \vec{A} \times (\vec{B} \times \vec{C}) = \hat{i} [A_y (B_x C_y - B_y C_x) - A_z (B_z C_x - B_x C_z)]$$

$$+ \hat{j} [A_x (B_y C_z - B_z C_y) - A_z (B_x C_y - B_y C_x)] + \hat{k} [A_x (B_z C_x - B_x C_z) - A_y (B_y C_z - B_z C_y)] \quad \dots\dots(1)$$

Now we evaluate $(\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$.

$$= (A_x C_x + A_y C_y + A_z C_z) (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) - (A_x B_x + A_y B_y + A_z B_z) (\hat{i}C_x + \hat{j}C_y + \hat{k}C_z)$$

$$= A_x B_x C_x \hat{i} + A_x B_y C_x \hat{j} + A_x B_z C_x \hat{k} + A_y B_x C_y \hat{i} + A_y B_y C_y \hat{j} + A_y B_z C_y \hat{k} + A_z B_x C_z \hat{i}$$

$$+ A_z B_y C_z \hat{j} + A_z B_z C_z \hat{k} - A_x B_x C_x \hat{i} - A_x B_x C_y \hat{j} - A_x B_x C_z \hat{k} - A_y B_y C_x \hat{i} - A_y B_y C_y \hat{j}$$

$$- A_y B_y C_z \hat{k} + A_z B_z C_x \hat{i} - A_z B_z C_y \hat{j} - A_z B_z C_z \hat{k}$$

$$\begin{aligned}
 & \text{i.e., } (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \\
 &= \hat{i}A_x B_x C_y - \hat{i}A_y B_y C_x + \hat{i}A_y B_x C_y - \hat{i}A_x B_y C_x + \hat{j}A_x B_y C_z - \hat{j}A_y B_x C_z \\
 &\quad + \hat{j}A_x B_y C_x - \hat{j}A_y B_x C_y + \hat{k}A_x B_z C_x - \hat{k}A_x B_x C_z + \hat{k}A_y B_z C_y - \hat{k}A_y B_y C_z \\
 & \text{i.e., } (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} = \hat{i}[A_y(B_x C_y - B_y C_x) - A_x(B_z C_x - B_x C_z)] \\
 &+ \hat{j}[A_x(B_y C_x - B_x C_y) - A_y(B_z C_x - B_x C_z)] + \hat{k}[A_x(B_z C_x - B_x C_z) - A_y(B_y C_z - B_z C_y)] \dots\dots\dots (2)
 \end{aligned}$$

From equation (1) and (2) we get,

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}. \quad (\text{Proved})$$

As an aid for memory we write

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\text{Dot product of first and third vector}) \text{ second vector} - (\text{Dot product of first and second vector}) \text{ third vector.}$$

Characteristics of Vector Triple Product :

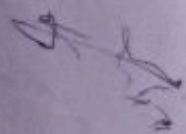
1. It is not commutative i.e., $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{B} \times \vec{C}) \times \vec{A}$ as $\vec{A} \times (\vec{B} \times \vec{C}) = -(\vec{B} \times \vec{C}) \times \vec{A}$
 (From property of cross product or vector product of two vectors).

2. It is not associative i.e., $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$.

As we know $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

But $(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{C} \times (\vec{A} \times \vec{B}) = -[(\vec{B} \cdot \vec{C})\vec{A} - (\vec{A} \cdot \vec{C})\vec{B}]$
 $= (\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A}$.

3. Let $\vec{D} = \vec{B} \times \vec{C}$ so \vec{D} is perpendicular to the plane containing \vec{B} and \vec{C} . Now $\vec{A} \times (\vec{B} \times \vec{C})$ is a vector perpendicular to plane formed by \vec{A} and $\vec{D}(\vec{B} \times \vec{C})$. So $\vec{A} \times (\vec{B} \times \vec{C})$ will be in the plane formed by \vec{B} and \vec{C} and in this plane it is perpendicular to \vec{A} .



Example - 1 : A parallelopiped has one vertex at the origin and has three adjacent vertices at $(10, -5, 3)$, $(3, -4, 7)$ and $(-5, -6, 3)$ in rectangular cartesian coordinates. Find the volume of the parallelopiped.

Answer : Let, $\vec{A} = 10\hat{i} - 5\hat{j} + 3\hat{k}$, $\vec{B} = 3\hat{i} - 4\hat{j} + 7\hat{k}$, $\vec{C} = -5\hat{i} - 6\hat{j} + 3\hat{k}$ as one vertex is at origin $(0,0,0)$

The volume of the parallelopiped is $\vec{A} \cdot (\vec{B} \times \vec{C})$.

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} 10 & -5 & 3 \\ 3 & -4 & 7 \\ -5 & -6 & 3 \end{vmatrix}$$

$$\text{i.e., Volume} = 10[-12 + 42] + 5[9 + 35] + 3[-18 - 20] \\ = 300 + 220 - 114 = 406 \text{ cubic units.}$$

Example - 2 : Prove that vectors $\hat{i} - 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar.

Answer : If value of $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$ then vectors are coplanar.

Let $\vec{A} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{B} = -2\hat{i} + 3\hat{j} - 4\hat{k}$, $\vec{C} = \hat{i} - 3\hat{j} + 5\hat{k}$

$$\text{Then, } \vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 1[15 - 12] + 2[-10 + 4] + 3[6 - 3] = 3 - 12 + 9 = 0.$$

So, the vectors are coplanar.

Example - 3 : Find the value of $[\vec{A} + \vec{B}, \vec{B} + \vec{C}, \vec{C} + \vec{A}]$

Answer : Let $\vec{P} = \vec{A} + \vec{B}$, $\vec{Q} = \vec{B} + \vec{C}$, $\vec{R} = \vec{C} + \vec{A}$

$$\therefore [\vec{A} + \vec{B}, \vec{B} + \vec{C}, \vec{C} + \vec{A}] = [\vec{P}, \vec{Q}, \vec{R}] = \vec{P} \cdot (\vec{Q} \times \vec{R}) \quad \dots(1)$$

$$\therefore \vec{Q} \times \vec{R} = (\vec{B} + \vec{C}) \times (\vec{C} + \vec{A}) = \vec{B} \times \vec{C} + \vec{B} \times \vec{A} + \vec{C} \times \vec{C} + \vec{C} \times \vec{A}$$

13. Show that $(\vec{A} \times \vec{B})$ is perpendicular to \vec{A} .

14. Prove that $[\vec{A} \times \vec{B}, \vec{B} \times \vec{C}, \vec{C} \times \vec{A}] = [\vec{A} \vec{B} \vec{C}]^2$.

15. Find the value of a such that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + a\hat{j} + 5\hat{k}$ are coplanar.

Ans. - 4

16. Find the unit vector which is perpendicular to $l\hat{i} + m\hat{j} + n\hat{k}$ and lies in xy plane.

Ans. $\pm \frac{1}{\sqrt{l^2 + m^2}}(m\hat{i} - l\hat{j})$

17. Show that vectors $-6\hat{i} + 9\hat{j} - 12\hat{k}$ and $2\hat{i} - 3\hat{j} + 4\hat{k}$ are parallel.

18. Find the angle between diagonal of a cube and one of the edges it meets at a vertex.

Ans. $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

19. Find the value of m and n so that $(2\hat{i} + 3\hat{j} - 5\hat{k})$, $(m\hat{i} - \hat{j} + 3\hat{k})$ and $(\hat{i} + \hat{j} + n\hat{k})$ are collinear.

20. A vector \vec{A} is turned through angle θ about the initial point. Prove that the magnitude of change in vector is $2A \sin \frac{\theta}{2}$.

Scalar and Vector Fields :

When a physical quantity is expressed as a continuous function of the position of a point in region of space, then the quantity is called point function. The region in which the physical quantity is specified is called as a field.

Scalar Field : It is defined as the region of space, such that each point of it, a scalar point function is linked. It is represented by a continuous function which represents value of the scalar quantity. It is expressed as $\phi(x, y, z)$ or $f(x, y, z)$ which is real and depends only on the point $P(x, y, z)$ in space, but not on any particular coordinate system used. Some examples of scalar fields are distribution of temperature on a heated body, Electric potential, magnetic potential, gravitational potentials etc, in a region of space.

Level Surface : If $f(x, y, z)$ is a single valued continuous scalar function defined at each point P in a domain, then $f(x, y, z) = c$ a constant defines the equation of a surface called level surface of the function. Ex if $f(x, y, z)$ represents temperature in a region then $f(x, y, z) = \text{constant}$ represents the surface at which temperature remains constant. This is called isothermal surface. Similarly equipotential surfaces are also level surfaces.

Vector Fields : It is represented by a vector function which is continuous and at any point it is specified by a vector of certain magnitude and direction. The magnitude and direction may change continuously from point to point throughout the vector field. In this field a vector function $\vec{F}(x, y, z)$ is represented by a vector, Examples are - Electric field, Magnetic field, Gravitational field etc.

